



Meshless Methods

Soheil Mohammadi

Professor of Computational Mechanics

School of Civil Engineering

University of Tehran

Tel: +98-21-6111 2258, Fax: +98-21-6640 3808, Cell: +98-912-5030 392

Email: smoham@ut.ac.ir, <http://eng.ut.ac.ir/civ/smohammadi>

Partition of Unity (PU)

1. Shepard Method (1968)
2. McLain Method (1974-1976)
3. Franke-Nielson (1980)
4. Duarte-Oden (HP-Clouds) (1995-1996)
5. Malenk-Babuska (Partition of Unity) (1995-1997)
6. Krysl-Belytschko (Linear Precision PU) (1999)
7. Many new contributions (...2005)

Shepard's Method (1968)

Assumptions:

- A set of scattered nodes is considered.
- Each node affects the approximation in its neighborhood (domain of influence)
- A weight function $w_I(x)$ is associated with each node :
 - Non-negative inside Ω_I
 - Vanishes on the boundary $\partial\Omega_I$
 - Non-zero at node I

Shepard approximation $u_h(x)$:

$$u_h(x) = \sum_{I=1}^N w_I(x) u_I$$

- u_I nodal parameters
- $W_I(x)$ weight function associated with the nodes
- $w_I(x)$ basis functions of compact support

$$w_I(x) = \frac{W_I(x)}{\sum_{K=1}^N W_K(x)}$$

Properties:

1. $w_I(x)$ functions represent a partition of unity

$$\sum_{I=1}^N w_I(x) = 1$$

2. A constant function $u(x) = c$ is reproduced exactly (constant precision property)

$$u_h(x) = \sum_{I=1}^N w_I(x) u_I = \sum_{I=1}^N w_I(x) c = c \sum_{I=1}^N w_I(x) = c$$

Properties:

3. The Shepard basis has only constant precision.
As a result, the PU Shepard basis does not converge for second-order PDEs.
It displays “flat-spots” at the nodes.
4. Other properties depend on the type of the weight function.

Franke-Nielson Weight Function:

- singular F-N weight function with compact support:

$$W_I(x_I, x) = \begin{cases} \left[\frac{(R_w - \|x - x_I\|)}{R_w \|x - x_I\|} \right]^2 & \text{for } \|x - x_I\| < R_w \\ 0 & \text{elsewhere} \end{cases}$$

R_w = radius of the support

Properties of w_I (Franke-Nielson):

1- Kronecker delta property

$$w_I(x_K) = \delta_{IK}$$

2- derivative

$$\frac{\partial w_I(x_K)}{\partial x_m} = 0, \quad m = 1, 2, \dots$$

3- If interpolation is not required, a non-singular weight function can be used

$$W_I(x) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & 1 > r \geq 0 \\ 0 & r \geq 1 \end{cases}$$

$$r = \frac{\|x - x_I\|}{R_w}$$

Modified Shepard Method:

- proposed by Franke and Nielson
- an inverse distance weighted least square interpolation
- the basic idea is to replace the nodal values u_I by a local approximating function $V_I(x)$ (also called a local fit or local function)

$$u_h(x) = \sum_{I=1}^N w_I(x) V_I(x)$$

$w_I(x)$ is the PU basis function

Modified Shepard Method:

1. The resulting approximation is a ***blend*** of the functions $V_I(x)$ through the PU basis $w_I(x)$
2. The span of this space can be chosen
 1. to fit the expected behaviour of the solution
 2. to reproduce polynomials of certain degree
3. The modified Shepard approximation reproduces exactly any function $f(x)$ contained in all the $V_I(x)$ functions.

$$u_h(x) = \sum_{I=1}^N w_I(x) V_I(x) = \sum_{I=1}^N w_I(x) f(x) = f(x) \sum_{I=1}^N w_I(x) = f(x)$$

Modified Shepard Method:

4. If singular F-N weight function is used:
 1. Modified shepard method interpolates at the nodes
 2. Derivatives of the approximation at the nodes are equal to the derivatives of the nodal functions $V_I(x)$
5. Shepard proposed using the derivative data (linear terms of a Taylor series) to achieve **linear precision** of the approximation basis

Modified Shepard Method:

6. Franke-Nielson proposed a quadratic local least-square fit:

$$V_I(x) = c_1^{(I)}(x - x_I)^2 + c_2^{(I)}(x - x_I)(y - y_I) + c_3^{(I)}(y - y_I)^2 \\ + c_4^{(I)}(x - x_I) + c_5^{(I)}(y - y_I) + c_6^{(I)}$$

1. If the PU function $w_I(x)$ allow for interpolation, the coefficient $c_6^{(I)}$ can be identified with the value of the approximated function at the node I ($c_6^{(I)} = u_I$), otherwise it needs to be determined by a weighted least square fit.

Modified Shepard Method:

2. The coefficients $c_j^{(I)}$, $j = 1, \dots, 5$ correspond to the derivatives of the solution (which is a major disadvantage)
3. The nodal parameters, therefore, have different physical dimensions.
4. The number of degrees of freedom per node increases

Linear Nodal PU Approximation:

- proposed by Belytschko
- provides linear precision
- uses only one type of nodal parameter

$$V_I(x) = \hat{\mu}_I^{(I)}(x)u_I + \sum_{K=1}^{N_I} \hat{\mu}_K^{(I)}(x)u_K$$

u_I = the nodal parameter associated with the node I

u_K = the nodal parameter associated with some other nodes which are ‘close’ to node I (star nodes)

$\hat{\mu}_m^{(I)}(x)$ = a set of linear functions

Linear Nodal PU Approximation:

$$V_I(x) = \hat{\mu}_I^{(I)}(x)u_I + \sum_{K=1}^{N_I} \hat{\mu}_K^{(I)}(x)u_K$$

1. The selection of the star nodes is quite arbitrary.
2. The star nodes are not required to be in the domain of influence of the node I .
3. Node I is not required to be in the domains of the influence of the star nodes.
4. Since the star nodes are essential in determining the derivative data, the closest nodes to node I can be expected to work best.

Linear Nodal PU Approximation:

$$V_I(x) = \hat{\mu}_I^{(I)}(x)u_I + \sum_{K=1}^{N_I} \hat{\mu}_K^{(I)}(x)u_K$$

5. If the singular weights are used, interpolation may be achieved if the nodal function $V_I(x)$ interpolates at the node I for any values of nodal parameters at the nodes of the nodal function.
6. If the nodal parameters are set to values of a linear polynomial function at the nodes, the nodal function V should interpolate at all the nodes contained by the V_I .
7. As a result, in this case, the nodal parameters are the values of the linear function at the nodes. (linear consistency condition)

Linear Nodal PU Approximation:

$$V_I(x) = \hat{\mu}_I^{(I)}(x)u_I + \sum_{K=1}^{N_I} \hat{\mu}_K^{(I)}(x)u_K$$

$$V_I(x) = u_I + \sum_{K=1}^{N_I} \mu_K^{(I)}(x)(u_K - u_I)$$

$$V_I(x) = u_I(1 - \sum_{K=1}^{N_I} \mu_K^{(I)}(x)) + \sum_{K=1}^{N_I} \mu_K^{(I)}(x)u_K$$

$\mu_K^{(I)}(x)$ is associated with node K in the star of node I .

Linear Nodal PU Approximation:

$$u_h(x) = \sum_{I=1}^N w_I(x) V_I(x)$$

$$u_h(x) = \sum_{I=1}^N w_I(x) \left[u_I \left(1 - \sum_{K=1}^{N_I} \mu_K^{(I)}(x) \right) + \sum_{K=1}^{N_I} \mu_K^{(I)}(x) u_K \right]$$

Linear Nodal PU Approximation:

$$u_h(x) = \sum_{I=1}^N \phi_I(x) u_I$$

$\phi_I(x)$ = the linear precision basis function

$$\phi_I(x) = w_I(x) \left(1 - \sum_{K=1}^{N_I} \mu_K^{(I)}(x) \right) + \sum_{K=1}^{A_I} w_K(x) \mu_I^{(K)}(x)$$

N = number of nodes in the influence domain

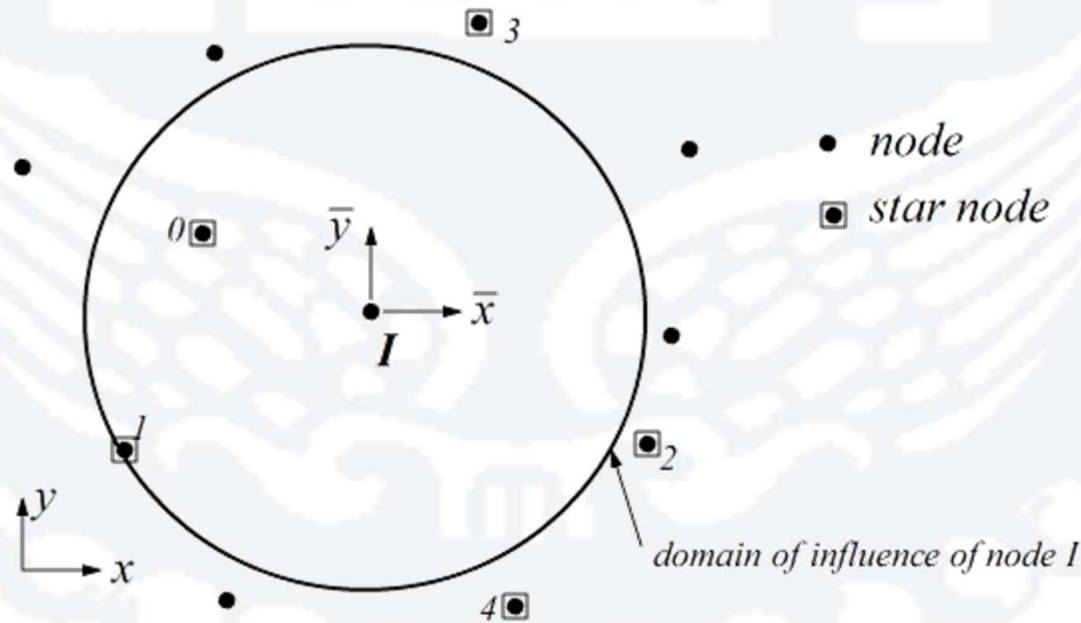
N_I = number of star nodes

A_I = number of star nodes in which the node I participates

Linear Nodal PU Approximation:

Local Basis Functions:

- 2D setting only.
- $(\bar{x}, \bar{y})$ Cartesian coordinate system parallel to global (x, y) and its origin at node I .



Linear Nodal PU Approximation:

Local Basis Functions:

- For the nodal function V_I to interpolate at node I , the linear function $\mu_K^{(I)}(x)$ is constructed so as to vanish at node I :

$$\mu_K^{(I)}(x) = \mu_K^{(I)}(\bar{x}) = a_K^{(I)} \frac{\bar{x}}{d_K} + b_K^{(I)} \frac{\bar{y}}{d_K}$$

$$d_K(x) = \|\bar{x}_K\| = \text{the distance of the star node K from node I (making } a_K^{(I)}, b_K^{(I)} \text{ non-dimensional)}$$

- In order to fully specify $V_I(x)$, we need to find the coefficients $a_K^{(I)}, b_K^{(I)}$ for each star node $K = 1, \dots, N_I$.

Linear Nodal PU Approximation:

Local Basis Functions:

- To construct a basis of linear precision, which reproduces **linear function exactly**, the nodal functions need to be able to reproduce exactly any complete linear polynomial $f(x) = \alpha x + \beta y + \gamma$
- or in local bases,

$$f(x) = A\bar{x} + B\bar{y} + C$$

where

$$\bar{x} = x - x_I$$

$$\bar{y} = y - y_I$$

and

$$A = \alpha$$

$$B = \beta$$

$$C = \gamma + \alpha x_I + \beta y_I$$

Linear Nodal PU Approximation:

Local Basis Functions:

- The nodal functions $V_I(x)$ are constructed so as to interpolate at node I and at all the star nodes for nodal parameters corresponding to a linear polynomial,

$$u_m = f(x_m) = A\bar{x}_m + B\bar{y}_m + C$$

Linear Nodal PU Approximation:

Local Basis Functions:

$$V_I(x) = f(x) = A\bar{x} + B\bar{y} + C$$

$$V_I(x) = f(x_I)(1 - \sum_{K=1}^{N_I} \mu_K^{(I)}(x)) + \sum_{K=1}^{N_I} \mu_K^{(I)}(x) f(x_K)$$

$$\bar{x}_I = \bar{y}_I = 0 \quad \therefore$$

$$V_I(x) = C(1 - \sum_{K=1}^{N_I} \mu_K^{(I)}(x)) + \sum_{K=1}^{N_I} \mu_K^{(I)}(x)(A\bar{x}_K + B\bar{y}_K + C)$$

$$V_I(x) = C + \sum_{K=1}^{N_I} \mu_K^{(I)}(x)(A\bar{x}_K + B\bar{y}_K)$$

Linear Nodal PU Approximation:

1. Constant precision is guaranteed.
2. Linear precision is achieved if:

$$\begin{cases} V_I(x) = A\bar{x} + B\bar{y} + C \\ V_I(x) = C + \sum_{K=1}^{N_I} \mu_K^{(I)}(x)(A\bar{x}_K + B\bar{y}_K) \end{cases}$$

$$\begin{cases} \sum_{K=1}^{N_I} a_K^{(I)} \left(A \frac{\bar{x}_K}{d_K} + B \frac{\bar{y}_K}{d_K} \right) = A \\ \sum_{K=1}^{N_I} b_K^{(I)} \left(A \frac{\bar{x}_K}{d_K} + B \frac{\bar{y}_K}{d_K} \right) = B \end{cases}$$

Linear Nodal PU Approximation:

3. and for arbitrary A and B ,

$$\begin{cases} \sum_{K=1}^{N_I} a_K^{(I)} \left(A \frac{\bar{x}_K}{d_K} + B \frac{\bar{y}_K}{d_K} \right) = A \\ \sum_{K=1}^{N_I} b_K^{(I)} \left(A \frac{\bar{x}_K}{d_K} + B \frac{\bar{y}_K}{d_K} \right) = B \end{cases}$$

$$\begin{cases} \sum_{K=1}^{N_I} a_K^{(I)} \frac{\bar{x}_K}{d_K} = 1 \\ \sum_{K=1}^{N_I} a_K^{(I)} \frac{\bar{y}_K}{d_K} = 0 \end{cases}, \quad \begin{cases} \sum_{K=1}^{N_I} b_K^{(I)} \frac{\bar{x}_K}{d_K} = 0 \\ \sum_{K=1}^{N_I} b_K^{(I)} \frac{\bar{y}_K}{d_K} = 1 \end{cases}$$

Linear Nodal PU Approximation:

4. For linear precision, there must be at least two star nodes, so that two $a_K^{(I)}$ and two $b_K^{(I)}$ can be calculated.
5. If there are more than two star nodes, one can
 1. solve for two of the coefficients
 2. the values of the remaining $(N_I - 2)$ coefficients can be chosen arbitrarily.
6. Existence of a solution (uniqueness)
 1. for the coefficients $a_1^{(I)}$ and $a_2^{(I)}$, the star nodes 1 and 2 must not be located on a line radially emanating from the node I .
 2. generally, vectors $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$ must be linearly independent

Linear Nodal PU Approximation:

7. The coefficients $a_K^{(I)}$ and $b_K^{(I)}$ ($K = 3, \dots, N_I$) are used to determine the slope of the approximated function at the nodes.
8. How the coefficients $a_K^{(I)}$ be distributed to achieve a good approximation to the derivatives of a given function at the node I ?
9. Krysl/Belytschko (1999) proposed the following rule for the initial selection of $a_K^{(I)}$ for any irregular distribution of the star nodes (the same analogy for $b_K^{(I)}$)

$$a_K^{(I)} = \left(\sum_{m=1}^{N_I} \frac{\bar{x}_m^2}{d_m^2} \right)^{-1} \frac{\bar{x}_K}{d_K}$$

Linear Nodal PU Approximation:

Krysl/Belytschko proposal:

1. For symmetric arrangements of the star nodes, $a_K^{(I)}$ satisfies the conditions for linear precision (by construction),

$$\sum_{K=1}^{N_I} a_K^{(I)} \frac{\bar{x}_K}{d_K} = 1 \quad , etc.$$

2. If for an arbitrary arrangement of the star nodes either of the conditions for the linear precision are not satisfied exactly for the proposed estimated value, $a_1^{(I)}$ and $a_2^{(I)}$ are then obtained via the direct solution of the same conditions.

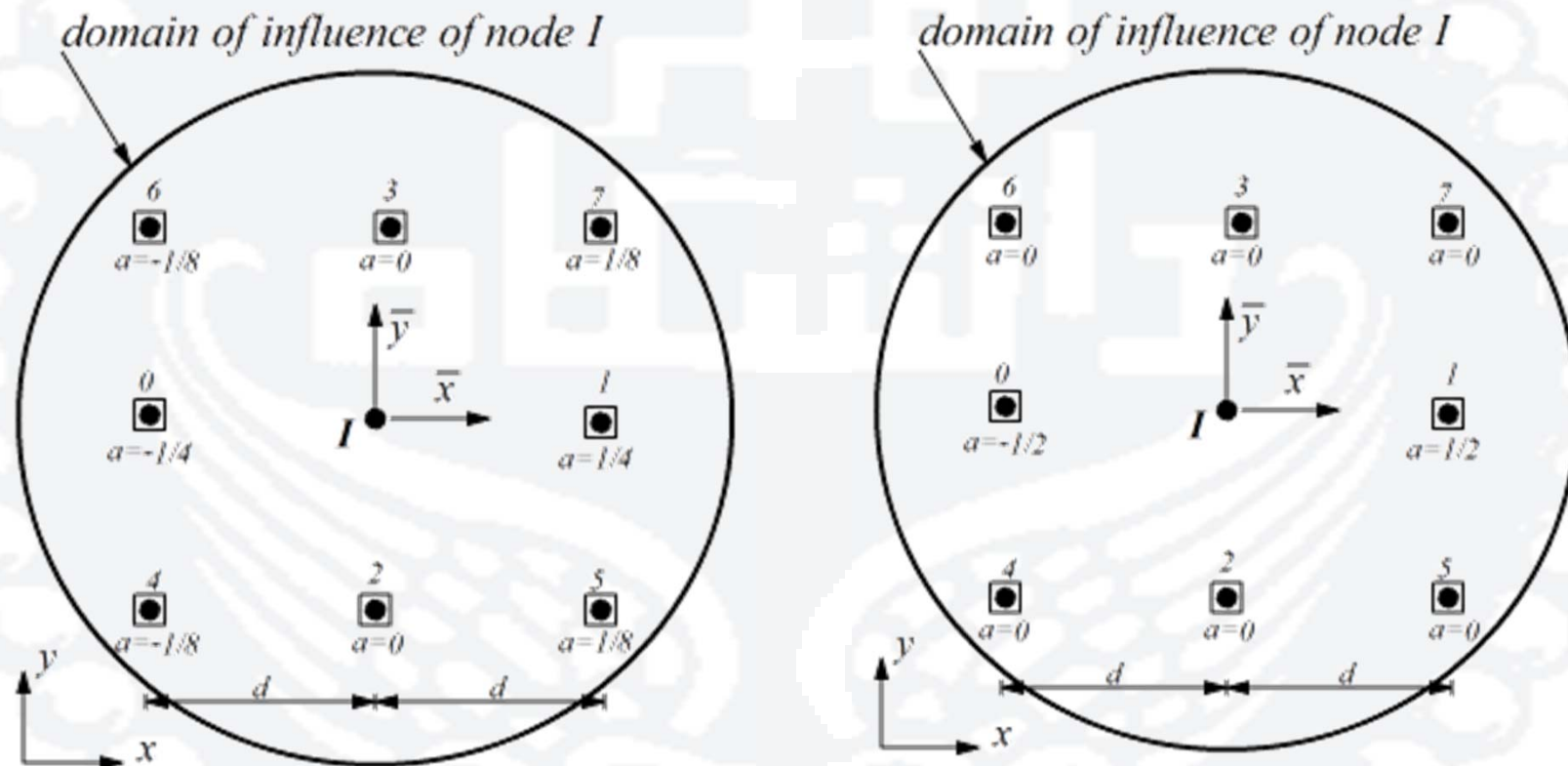
Linear Nodal PU Approximation:

Krysl/Belytschko proposal:

3. Deterioration of the absolute accuracy is expected for non-optimal distribution of the coefficients.
4. The convergence rate is not affected in any case, since the linear precision conditions have been already met.
5. Further investigation is required to determine if there are optimality criterion for the selection of the stars and for the distribution of the coefficients in the stars.

Linear Nodal PU Approximation:

Symmetry:



Linear Nodal PU Approximation:

Symmetry:

1. It is reasonable to require some kind of symmetry of the distribution with respect to the axes $\bar{x}, \bar{y}$.
2. For a perfectly symmetric distribution of nodes, the coefficients should be symmetric.
3. Both right and left distributions of coefficients $a_K^{(I)}, K = 0, \dots, 7$ are acceptable for the depicted star nodes .
4. The left side loosely corresponds to a central difference estimate of the first derivative with respect to x at the central node.

Linear Nodal PU Approximation:

Symmetry:(cont.)

5. Some co-relations can be made between the design of the nodal function in the present method and the finite difference approximations of derivative data, in particular to the generalised finite difference method (G-FDM).

Linear Nodal PU Approximation:

Numerical Examples:

1. Surface functions:

$$1. \quad \begin{cases} S(x, y) = (-x^3 + \sin(2y) + xy/2)/4 \\ -2 \leq x, y \leq 2 \quad , l = 4 \end{cases}$$

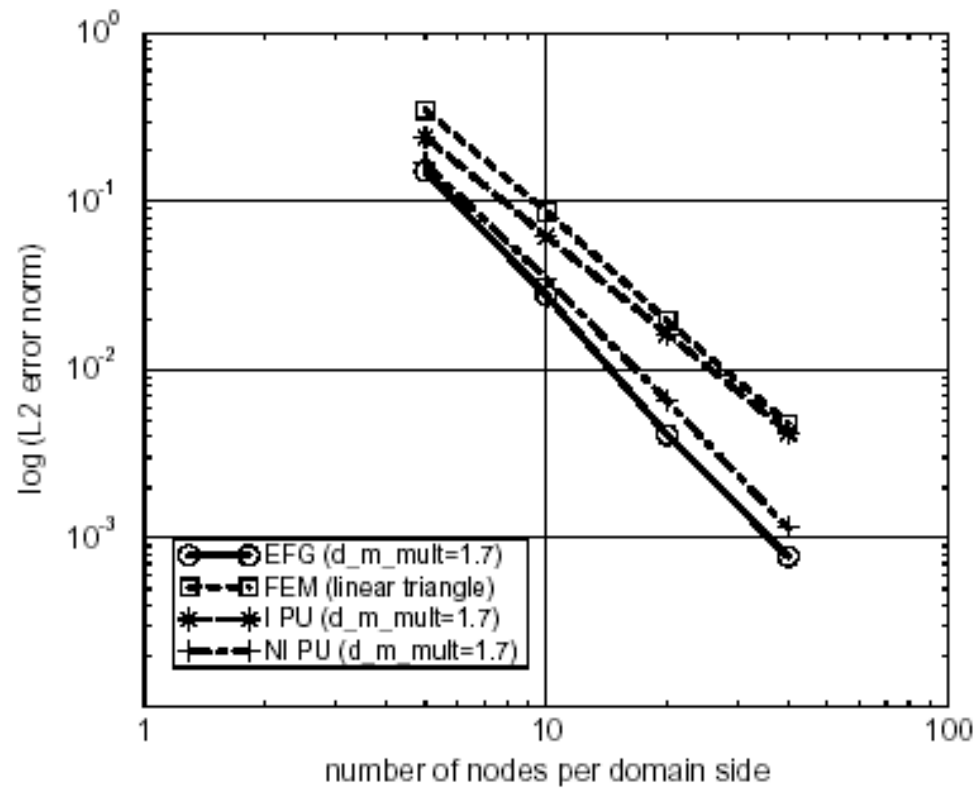
$$2. \quad \begin{cases} R(x, y) = -xy \exp(-2x^4 y^4) \\ -2 \leq x, y \leq 2 \quad , l = 4 \end{cases}$$

2. Four grids with spacings: $l/5, l/10, l/20, l/40$

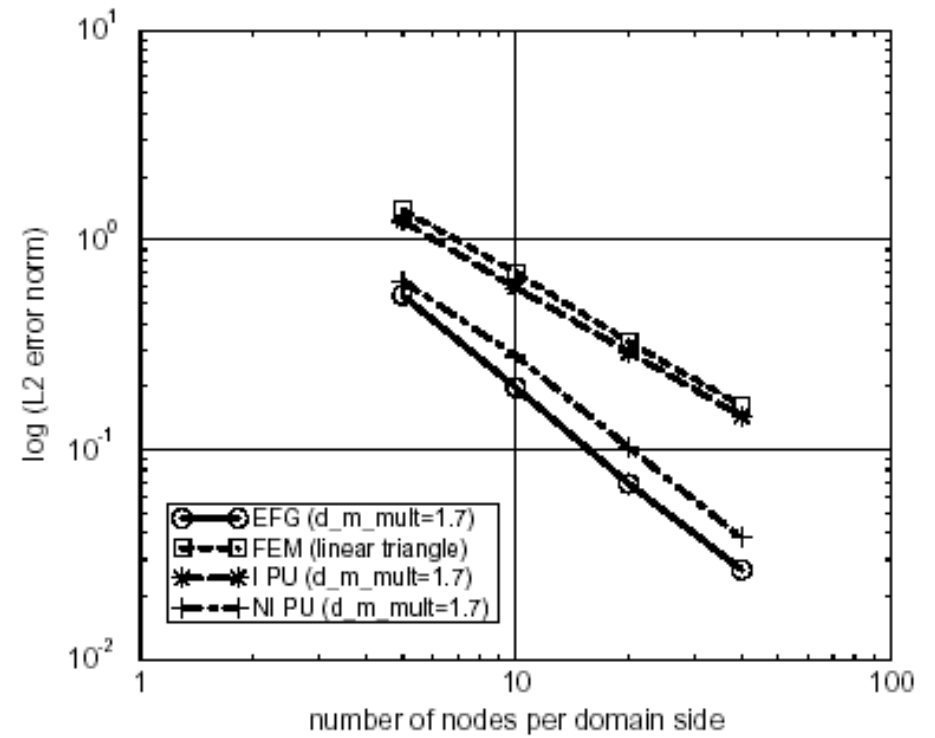
3. l_e is the average edge length in the triangulation

4. The support size was set to $1.7l_e$

$$S(x, y) = (-x^3 + \sin(2y) + xy/2) / 4$$

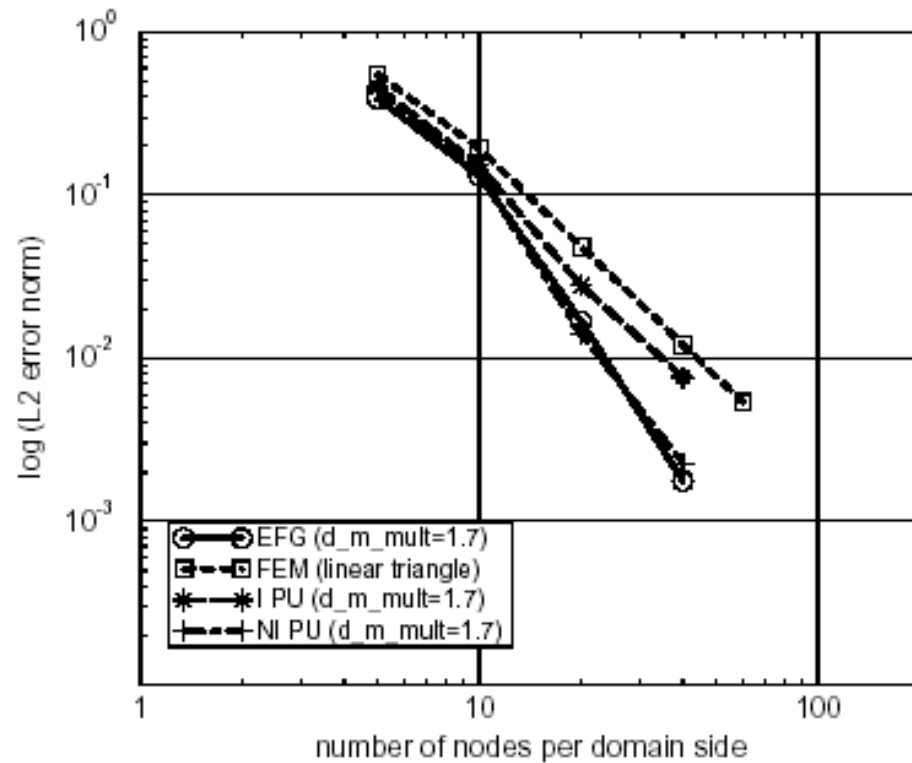


(a) Error $\|f - f_h\|_{L_2}$.

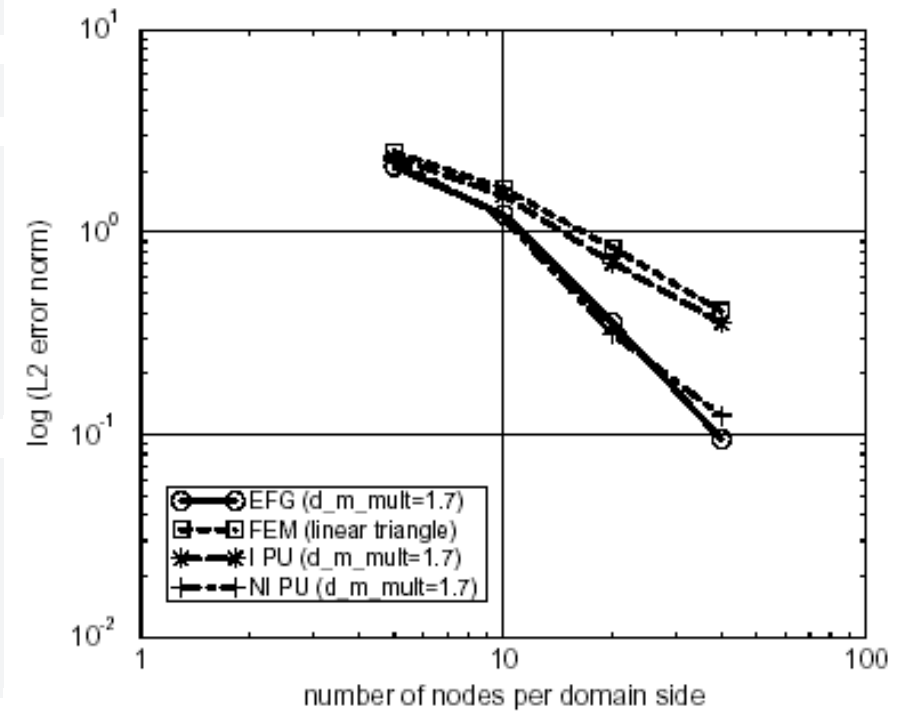


(b) Error $\|\partial f - \partial f_h\|_{L_2}$.

$$R(x, y) = -xy \exp(-2x^4 y^4)$$

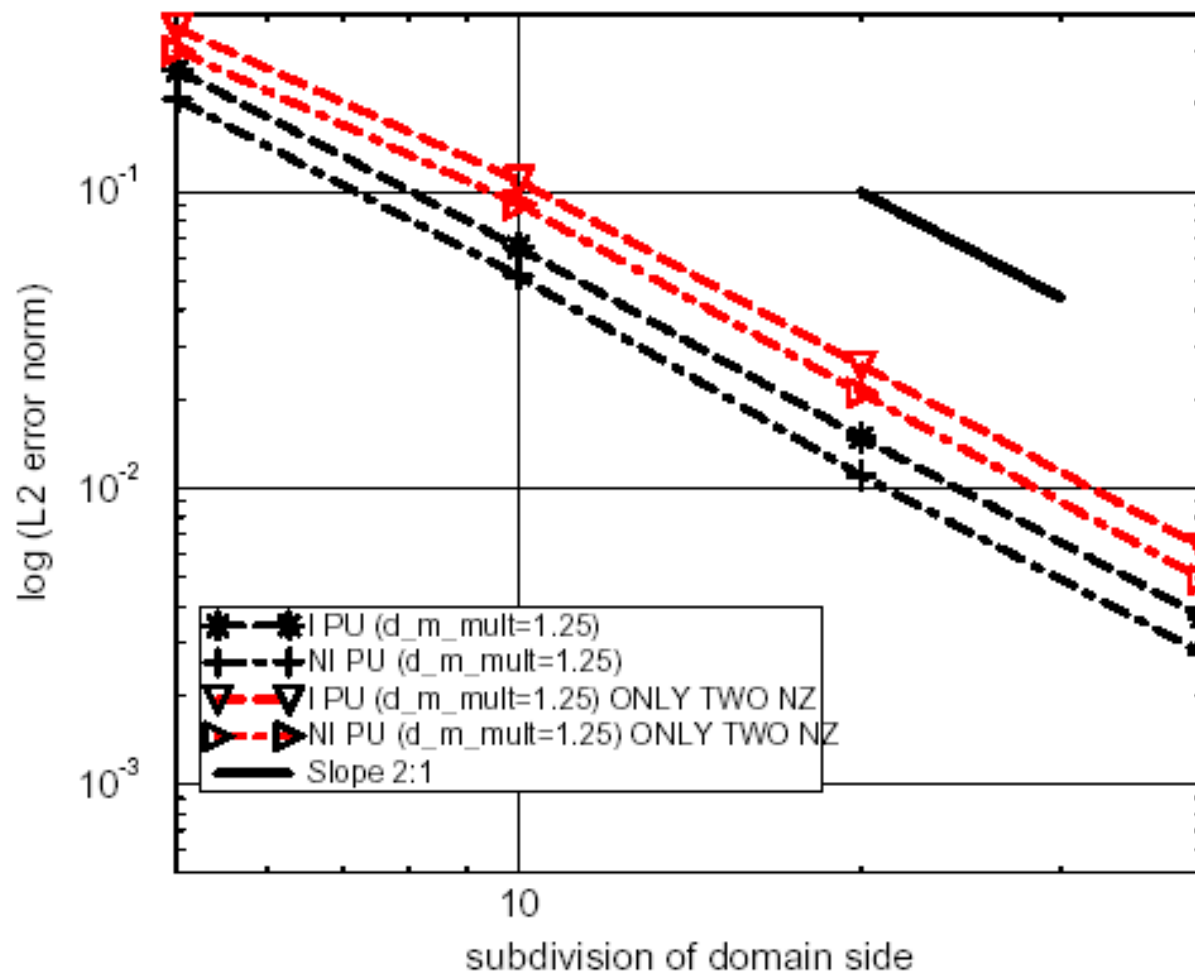


(a) Error $\|f - f_h\|_{L_2}$.



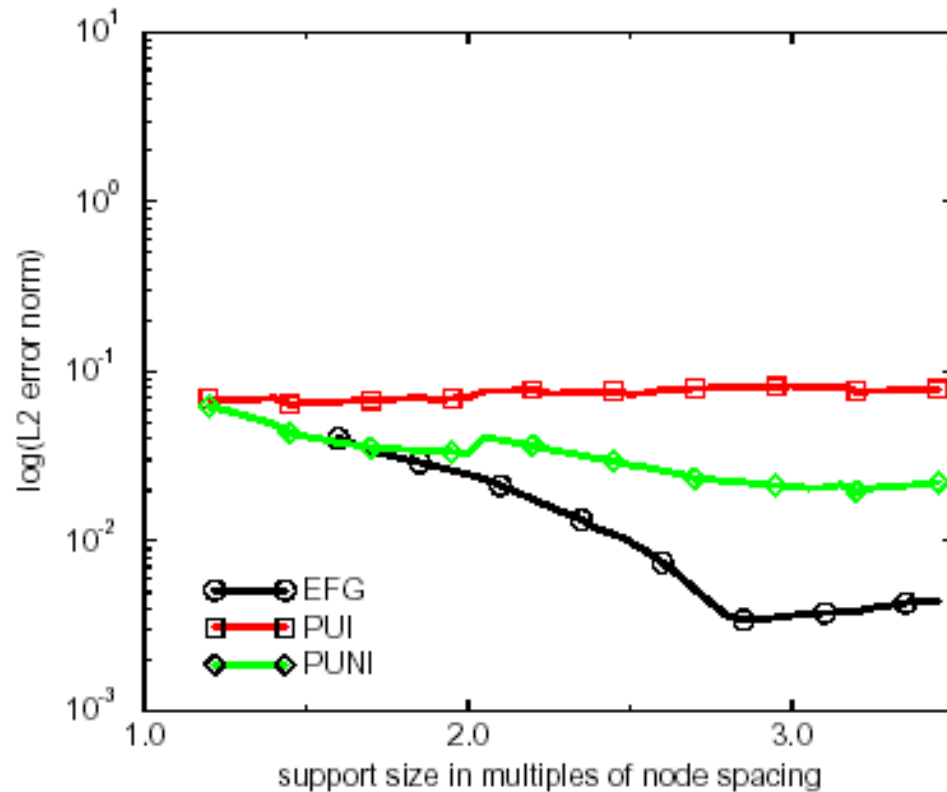
(b) Error $\|\partial f - \partial f_h\|_{L_2}$.

$$S(x, y) = (-x^3 + \sin(2y) + xy/2) / 4$$

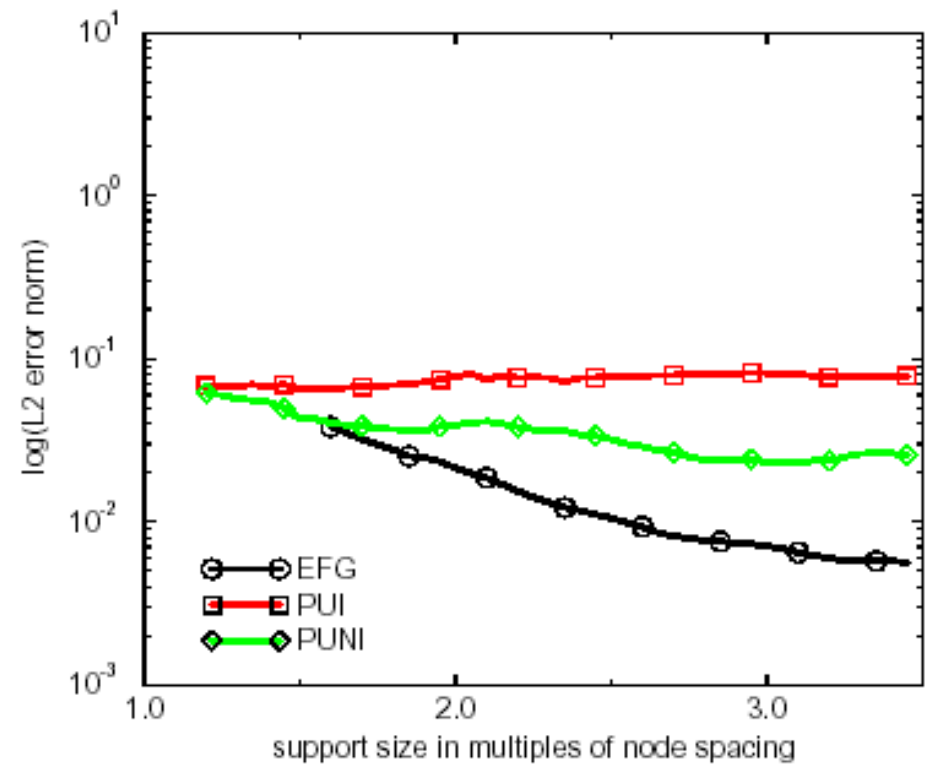


$$S(x, y) = (-x^3 + \sin(2y) + xy/2) / 4$$

Star nodes = all nodes within the domain of influence



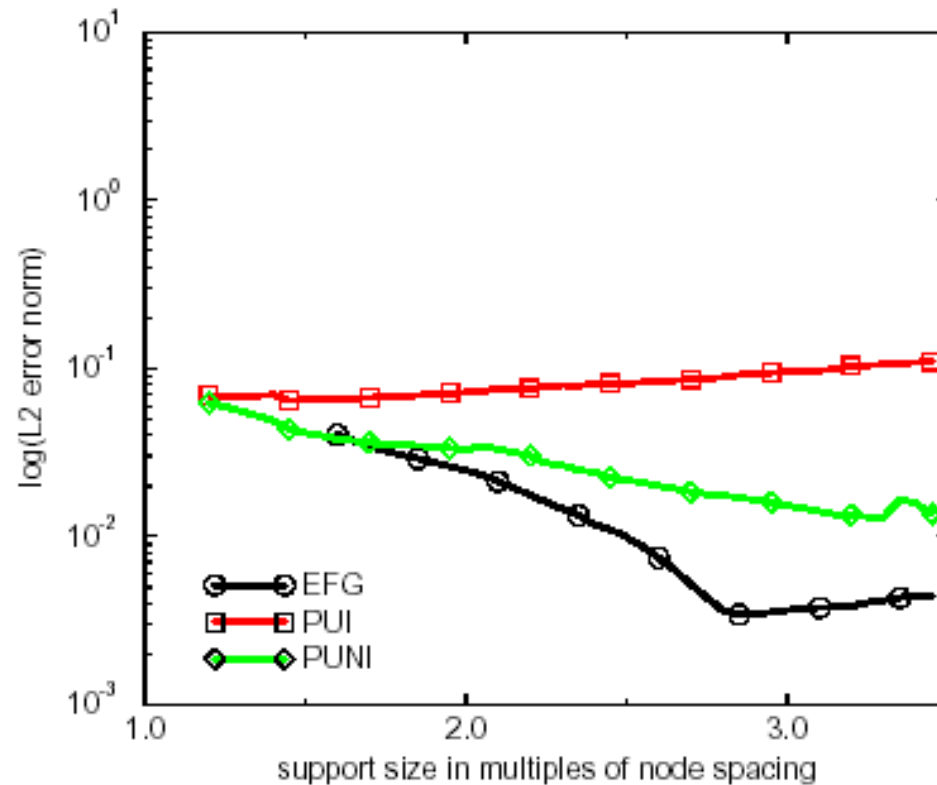
(a) Regular grid



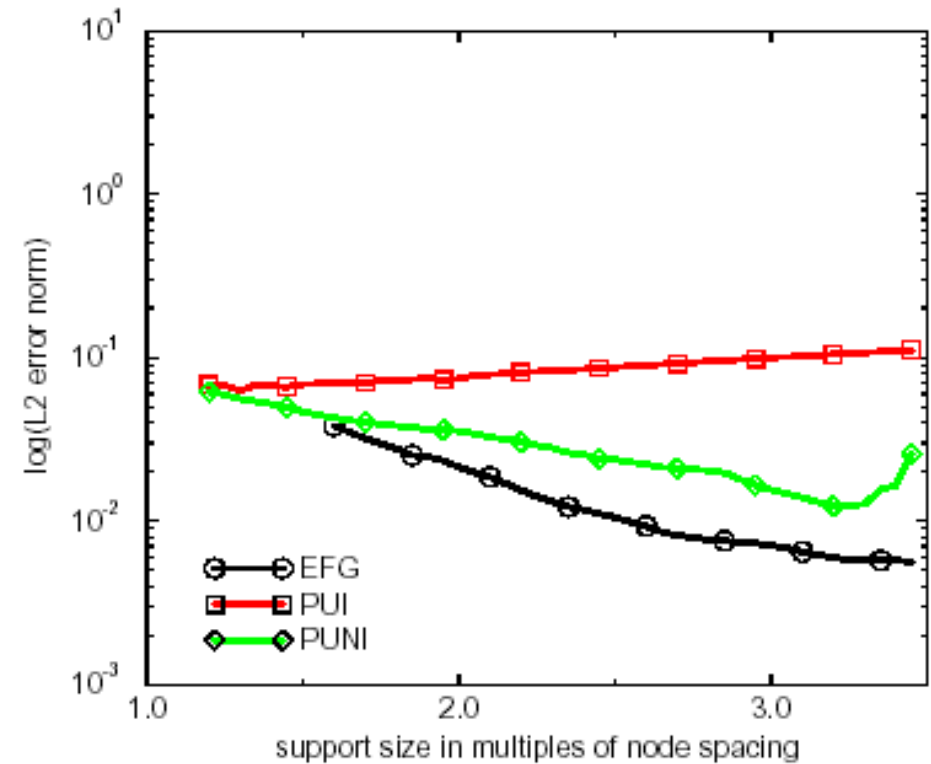
(b) Irregular grid

$$S(x, y) = (-x^3 + \sin(2y) + xy/2) / 4$$

Star nodes = the closest four nodes

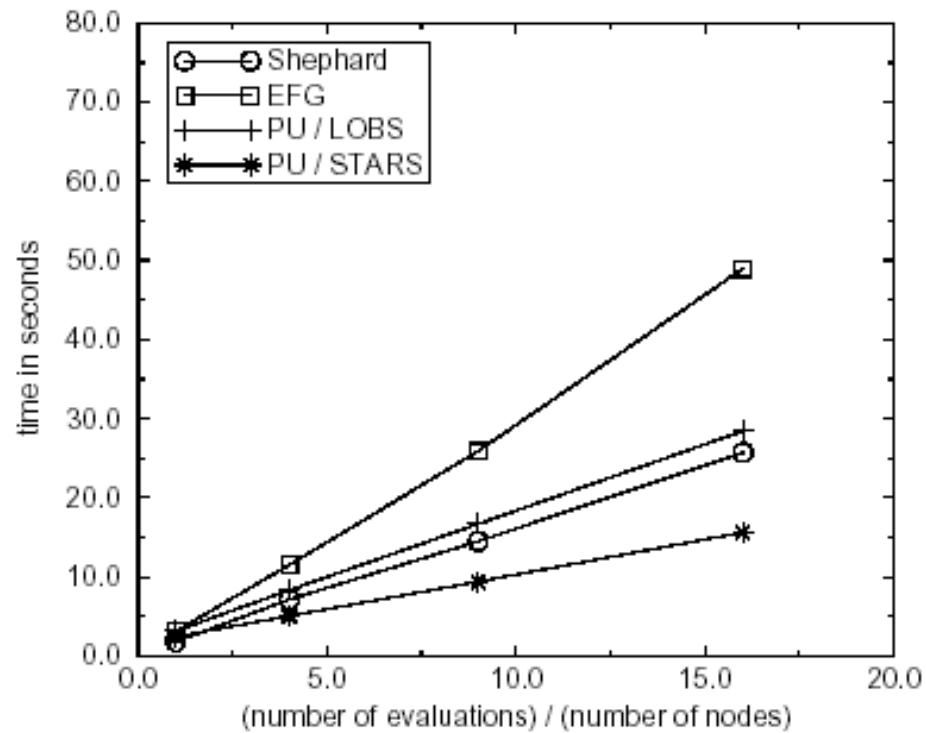


(a) Regular grid

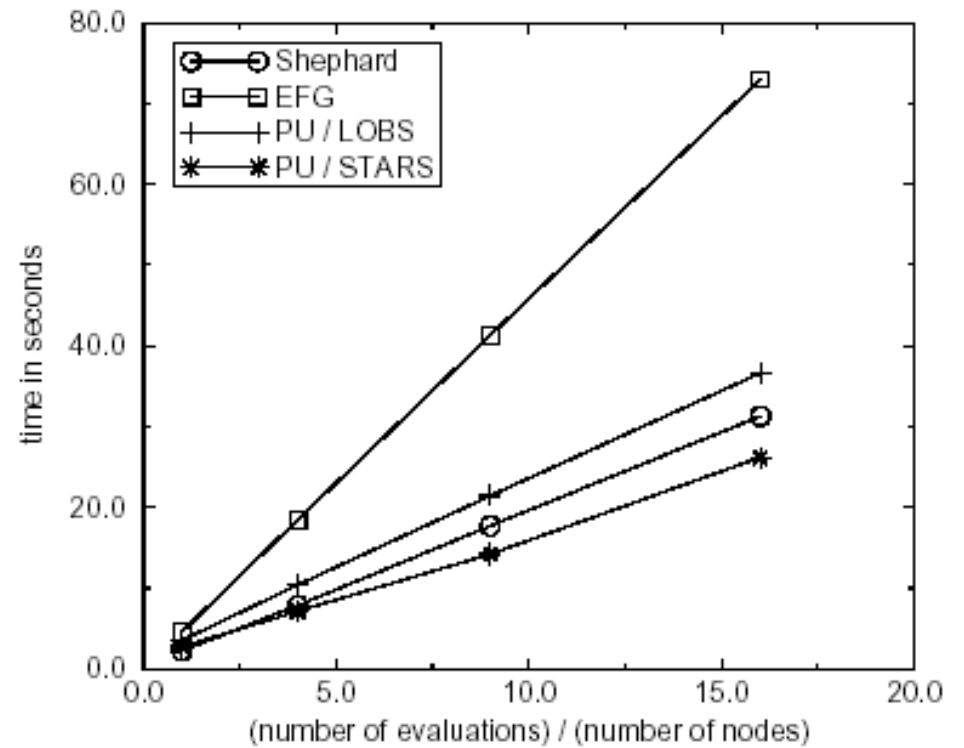


(b) Irregular grid

$$S(x, y) = (-x^3 + \sin(2y) + xy/2) / 4$$



(a) 12 neighbors



(b) 32 neighbors

Linear Nodal PU Approximation:

Summary:

1. The grid consists of scattered nodes
2. The basis exactly reproduces complete linear polynomials
3. Only the values of the approximated function at the nodes are used as unknowns
4. The construction of the basis is only slightly more expensive than the Shepard constant precision method
5. The method is applicable in any number of dimensions

